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Spin-Orbit Interaction from Matrix Theory

Per Kraus¹

*California Institute of Technology
Pasadena CA 91125, USA
email: perkraus@theory.caltech.edu*

Abstract:

We study the leading order spin dependence of graviton scattering in eleven dimensions, and show that the results obtained from supergravity and from Matrix Theory precisely agree.

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1 Introduction

There are by now a substantial number of checks of the correspondence between Matrix Theory [1] and eleven dimensional supergravity. Impressive though they are, these checks have probed only a very limited part of the structure of supergravity. Among the most striking of the checks are the successful computations of the v^4/r^7 and v^6/r^{14} terms arising in the scattering of two gravitons [1, 2, 3, 4]. The results can be thought of as probing the cubic and quartic interaction vertices of supergravity. However, these calculations effectively average over the polarizations of the scattered gravitons, and so are sensitive only to the magnitude of the vertices and not to their tensor structures. In this work we will try to see whether the tensor structure comes out correctly by studying the leading order spin dependence of graviton scattering.

There have recently been some string theory analyses of the spin dependence of D0-brane scattering, or equivalently, graviton scattering in compactified M theory. [5] used a series of duality transformations to map the problem to one involving fundamental strings, and [6] approached the problem through the boundary state formalism. Here we proceed somewhat differently, by finding the linearized metric of a spinning D0-brane and studying the the action of a D0-brane probe moving in that background. Since we will be working with the linearized theory, we will only pick up terms of first order in the spin, whereas [5, 6] found the higher order contributions as well. In the present approach, by solving the full field equations it should be straightforward to recover the extra terms, but we will not attempt that here.

Our approach has the advantage that it can be extended to include contributions which are of higher orders in the gravitational coupling. The problem with the other methods is that to compare with Matrix Theory beyond the lowest order one must consider not standard IIA theory, but rather the theory resulting from compactifying M theory along a null direction [7]. In our framework this can easily be implemented by lifting the solution to eleven dimensions and then recompactifying along a null direction. It is less clear how to proceed within the framework of [5, 6].

On the Matrix Theory side, we will compute one loop contributions to the effective action in the presence of both bosonic and fermionic background fields. The fermions encode the spin of the D0-brane in precisely the right way to reproduce the supergravity result. The term we compute is part of the supersymmetric completion of the bosonic $(F_{\mu\nu})^4$ terms which arise at one loop. It would be nice to demonstrate that supersymmetry is sufficient to fix the coefficients of the fermionic terms.

Before proceeding to the calculations, let us mention that in principle an efficient way of approaching the supergravity side of the problem would be through use of the supersymmetric Born-Infeld action [8, 9, 10]. The coupling of a supersymmetric D0-

brane to target space fields has been worked out in [9, 10]. However, the results there are presented in terms of superfields, whereas to apply them to the problem we are studying one really needs to work out their component forms.

2 Spin-orbit potential from supergravity

We would like to compute, in ten dimensions, the linearized metric produced by a D0-brane of mass T_0 and angular momentum J^{ij} . It is easiest to begin by considering the Einstein metric; formulas for the linearized metric are then found in [11]. Writing

$$g_{\mu\nu}^E = \eta_{\mu\nu} + h_{\mu\nu}^E,$$

with $\eta_{\mu\nu} = \text{diag}(-, +, + \cdots +)$, the needed formulas are

$$\begin{aligned} h_{00}^E &= \frac{16\pi G_N T_0}{8\omega_8 r^7} \\ h_{ij}^E &= \frac{16\pi G_N T_0}{56\omega_8 r^7} \delta_{ij} \\ h_{0i}^E &= -\frac{8\pi G_N x^k J^{ki}}{\omega_8 r^9}. \end{aligned} \tag{1}$$

where ω_8 is the area of the unit 8-sphere. It is convenient to introduce the quantity

$$Q_0 = \frac{16\pi G_N T_0}{7\omega_8} = \frac{15}{2T_0}.$$

Note that we are working in string units ($2\pi\alpha' = 1$). Then,

$$\begin{aligned} h_{00}^E &= \frac{7}{8} \frac{Q_0}{r^7} \\ h_{ij}^E &= \frac{1}{8} \frac{Q_0}{r^7} \delta_{ij} \\ h_{0i}^E &= -\frac{7}{2} \frac{Q_0}{T_0} \frac{x^k J^{ki}}{r^9}. \end{aligned} \tag{2}$$

Now we transform to the string metric using,

$$g_{\mu\nu}^S = e^{\phi/2} g_{\mu\nu}^E.$$

To linear order, the dilaton takes the same value as it does in the unspinning case,

$$e^{-\phi} = 1 - \frac{3}{4} \frac{Q_0}{r^7}.$$

Thus we find,

$$\begin{aligned}
g_{00}^S &= -\left(1 - \frac{Q_0}{2r^7}\right) \\
g_{ij}^S &= \left(1 + \frac{Q_0}{2r^7}\right) \delta_{ij} \\
g_{0i}^S &= -\frac{7}{2} \frac{Q_0}{T_0} \frac{x^k J^{ki}}{r^9}.
\end{aligned} \tag{3}$$

g_{00}^S, g_{ij}^S of course take their standard values, while g_{0i}^S gives the spin contribution.

Actually, the metric that we want is obtained from the one above by dropping the 1's in the parentheses of g_{00}^S, g_{ij}^S . This is because we want a solution corresponding to the theory obtained by compactifying eleven dimensional supergravity along a null direction. Such a solution can be generated by lifting the metric (3) to eleven dimensions using the standard formulas for spacelike compactification, and then returning to ten dimensions by compactifying a null direction. The result is precisely to remove the 1's just mentioned. In fact, this procedure is unnecessary in the present context, as it only affects terms of higher power in velocity than the spin-orbit term, but we mention it here for completeness. From now on we drop the 1's and refer to the resulting metric as simply $g_{\mu\nu}$.

Next, we consider the action of a D0-brane probe moving in this background:

$$S_0 = -T_0 \int dt \left\{ e^{-\phi} \sqrt{-g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu} - C_\mu \dot{X}^\mu \right\}. \tag{4}$$

To compute the potential we need to know the value of the RR gauge field C_μ . We take

$$C_0 = -\frac{Q_0}{r^7} \quad ; \quad C_i = \frac{7}{4} \frac{Q_0}{T_0} \frac{x^k J^{ki}}{r^9}.$$

The form of C_0 is the conventional one. The value for C_i could be arrived at by determining the magnetic moment of the D0-brane and using the standard formula for the resulting magnetic field. Instead we have chosen the coefficient 7/4 so as to cancel a term linear in velocity coming from the expansion of $\sqrt{-g \dot{X} \dot{X}}$. The linear term is known to be absent (from the calculations of [5, 6], for example). At any rate, this term is irrelevant as far as the coefficient of the spin-orbit term is concerned.

Now, inserting the fields into S_0 and expanding in powers of velocity we find,

$$\begin{aligned}
S_0 &= -T_0 \int dt \left\{ 1 - \frac{1}{2} \vec{v} \cdot \vec{v} + \frac{7}{8} \frac{Q_0}{T_0} \frac{(x^i J^{ij} v^j)(\vec{v} \cdot \vec{v})}{r^9} - \frac{Q_0}{8} \frac{(\vec{v} \cdot \vec{v})^2}{r^7} + \dots \right\} \\
&= \int dt \left\{ -T_0 + \frac{T_0}{2} \vec{v} \cdot \vec{v} - \frac{105}{16 T_0} \frac{(x^i J^{ij} v^j)(\vec{v} \cdot \vec{v})}{r^9} + \frac{15}{16} \frac{(\vec{v} \cdot \vec{v})^2}{r^7} + \dots \right\}.
\end{aligned} \tag{5}$$

The third term gives the spin-orbit term which we would like to reproduce from Matrix Theory.

3 Spin-orbit potential from Matrix Theory

On the Matrix Theory side the calculation proceeds by evaluating the quantum mechanical effective action of a system of two D0-branes. Before doing any explicit computation, dimensional analysis and the systematics of the loop expansion allow one to write the general form of the effective action [4, 12],

$$S_l \sim \int dt r^{4-3l} f\left(\frac{v}{r^2}, \frac{\psi}{r^{3/2}}\right). \quad (6)$$

Here S_l denotes the l 'th loop contribution. Note that at the one loop level the action includes the term $v^3\psi^2/r^8$, and that this term has the correct v and r dependence to match onto the spin-orbit term of (5). We would like to check whether the numerical coefficient and the tensor structure similarly agree.

We will be following the conventions of [3], and some details of the calculation which we omit can be found there. The Matrix theory action, including gauge fixing and ghost terms, is

$$S = \int dt \text{Tr} \left\{ \frac{T_0}{2} F_{\mu\nu} F^{\mu\nu} - i\bar{\psi} \not{D} \psi + T_0 (\bar{D}^\mu A_\mu)^2 \right\} + S_{\text{ghost}}, \quad (7)$$

where $\mu, \nu = 0 \dots 9$, $A_\mu = (A, X_i)$, and

$$\begin{aligned} \bar{D}^\mu A_\mu &= -\partial_t A + [B^i, X_i] \\ F_{0i} &= \partial_t X_i + [A, X_i] \\ F_{ij} &= [X_i, X_j] \\ D_t \psi &= \partial_t \psi + [A, \psi] \\ D_i \psi &= [X_i, \psi] \end{aligned} \quad (8)$$

B^i is the bosonic background field. The fluctuations about the background will be denoted by Y^i ,

$$X^i = B^i + \frac{Y^i}{\sqrt{T_0}}.$$

We will be studying a system of two D0-branes, so all fields take values in the Lie algebra of $U(2)$. In terms of the $U(2)$ generators we write,

$$\begin{aligned} A &= \frac{i}{2} (A_0 \mathbf{1} + A_a \sigma^a) \\ X^i &= \frac{i}{2} (X_0^i \mathbf{1} + X_a^i \sigma^a) \\ \psi &= \frac{i}{2} (\psi_0 \mathbf{1} + \psi_a \sigma^a) \end{aligned} \quad (9)$$

For the background fields, we will give nonzero values to

$$B_3^1 = vt, \quad B_3^2 = b \quad \text{and} \quad \psi_3.$$

The values given to B_a^i correspond to two D0-branes moving with relative velocity v along the x^1 direction, and separated by distance b along the x^2 direction.

The fermionic background ψ_3 gives fermionic expectation value $\pm\psi_3/2$ to each of the two D0-branes. However, one is free to shift the background by an amount proportional to $\mathbf{1}$ since the U(1) part of the action decouples from the SU(2) part. Thus the setup equally well applies to the case where the two D0-branes have fermionic expectation values ψ_3 and 0. The latter picture corresponds to the supergravity configuration we are trying to describe.

Now it is straightforward but tedious to expand out the action in terms of the field components defined above. We work in Euclidean space $t \rightarrow i\tau, A \rightarrow -iA$. The action takes the schematic form

$$S \sim (A)^2 + (Y)^2 + (\psi)^2 + \dot{B}AY + \psi_3 Y \psi + \psi_3 A \psi + \dots, \quad (10)$$

where \dots indicates terms cubic and quartic in fluctuations which won't contribute to our analysis. Our strategy will be to treat the first four terms exactly and the last two terms perturbatively. That is, in terms of Feynman diagrams, the first four terms supply the propagators and the last two will supply the vertices. Let us first study the mass spectrum by considering the propagator terms. One finds the following mass eigenstates:

$$\begin{aligned} Y_\pm^n &= \frac{Y_1^n \pm iY_2^n}{\sqrt{2}} \quad (n = 2 \dots 9) & m^2 &= r^2 \\ S_\pm &= \frac{Y_1^1 \pm iY_2^1 \mp iA_1 + A_2}{\sqrt{2}} & m^2 &= r^2 + 2v \\ T_\pm &= \frac{Y_1^1 \pm iY_2^1 \mp iA_1 - A_2}{\sqrt{2}} & m^2 &= r^2 - 2v \\ \psi_\pm &= \frac{\psi_1 \pm i\psi_2}{\sqrt{2}} & m &= v\tau\gamma_1 + b\gamma_2 \end{aligned} \quad (11)$$

where $r^2 = b^2 + (v\tau)^2$. Here, by “mass eigenstate” we mean that the action takes the form

$$i \int d\tau \frac{1}{2} \phi_+ (\partial_\tau^2 - m_\phi^2) \phi_- \quad \text{and} \quad i \int d\tau \psi_+^T (\partial_\tau - m_\psi) \psi_-$$

for the case of bosons and fermions respectively. In addition to the massive fields just described, there are massless fields which play no role in the following discussion.

Given these quadratic actions, we can work out propagators. For the bosonic fields we define $\Delta_B(\tau_1, \tau_2 | m^2)$ as the solution to

$$(-\partial_{\tau_1}^2 + m^2)\Delta_B(\tau_1, \tau_2 | m^2) = \delta(\tau_1 - \tau_2). \quad (12)$$

Note that m is allowed to be time dependent. Then we find,

$$\begin{aligned} \langle Y_-^n(\tau_1)Y_+^{n'}(\tau_2) \rangle &= \Delta_B(\tau_1, \tau_2 | r^2) \delta^{nn'} \\ \langle S_-(\tau_1)S_+(\tau_2) \rangle &= \Delta_B(\tau_1, \tau_2 | r^2 + 2v) \\ \langle T_-(\tau_1)T_+(\tau_2) \rangle &= \Delta_B(\tau_1, \tau_2 | r^2 - 2v). \end{aligned} \quad (13)$$

We similarly define Δ_F by

$$(-\partial_{\tau_1} + m)\Delta_F(\tau_1, \tau_2 | m) = \delta(\tau_1 - \tau_2). \quad (14)$$

Then

$$\langle \psi_+(\tau_1)\psi_-(\tau_2) \rangle = \Delta_F(\tau_1, \tau_2 | v\tau_1\gamma_1 + b\gamma_2). \quad (15)$$

In fact, Δ_F can be related to Δ_B by

$$\Delta_F(\tau_1, \tau_2 | v\tau_1\gamma_1 + b\gamma_2) = (\partial_{\tau_1} + v\tau_1\gamma_1 + b\gamma_2)\Delta_B(\tau_1, \tau_2 | r^2 - v\gamma_1). \quad (16)$$

It will turn out that we won't need the full structure of Δ_F , but only part of it. The formulas we will need are

$$\begin{aligned} \psi_3^T P_+ \Delta_F(v\tau_1\gamma_1 + b\gamma_2) P_- \psi_3 &= \frac{b}{2} \psi_3^T \gamma_1 \gamma_2 \psi_3 \Delta_B(r^2 + v) \\ \psi_3^T P_- \Delta_F(v\tau_1\gamma_1 + b\gamma_2) P_+ \psi_3 &= -\frac{b}{2} \psi_3^T \gamma_1 \gamma_2 \psi_3 \Delta_B(r^2 - v) \\ \sum_{n=2}^9 \psi_3^T \gamma_n \Delta_F(v\tau_1\gamma_1 + b\gamma_2) \gamma_n \psi_3 &= \frac{3b}{2} \psi_3^T \gamma_1 \gamma_2 \psi_3 [\Delta_B(r^2 + v) - \Delta_B(r^2 - v)], \end{aligned} \quad (17)$$

where $P_{\pm} = (1 \pm \gamma_1)/2$, and we have suppressed the τ dependence. These relations are easily derived upon recalling $\psi_3^T \psi_3 = \psi_3^T \gamma_i \psi_3 = 0$, which follows from the grassmann property of ψ_3 and the symmetry of γ_i .

Now we can work out the fermionic dependence of the one loop effective action. For this, we need to first find the $\psi_3 Y \psi$ and $\psi_3 A \psi$ terms in $S_{\text{fermi}} = -i \int dt \text{Tr } \bar{\psi} \not{D} \psi$. We find,

$$\begin{aligned} S_{\text{fermi}} &= -\frac{i}{\sqrt{T_0}} \int d\tau \left\{ Y_-^n \psi_3^T \gamma_n \psi_+ + Y_+^n \psi_-^T \gamma_n \psi_3 + \sqrt{2} S_- \psi_3^T P_+ \psi_+ - \sqrt{2} S_+ \psi_-^T P_- \psi_3 \right. \\ &\quad \left. - \sqrt{2} T_- \psi_3^T P_- \psi_+ + \sqrt{2} T_+ \psi_-^T P_+ \psi_3 \right\}. \end{aligned} \quad (18)$$

The spin-orbit interaction is found by expanding $e^{iS_{\text{fermi}}}$ to quadratic order in ψ_3 and taking the vacuum expectation value. This gives

$$S_{\text{so}} = -\frac{1}{T_0} \int d\tau_1 d\tau_2 \{ \langle Y_-^n(\tau_1) Y_+^{n'}(\tau_2) \rangle \psi_3^T \gamma_n \langle \psi_+(\tau_1) \psi_-(\tau_2) \rangle \gamma_{n'} \psi_3 \\ - 2 \langle S_-(\tau_1) S_+(\tau_2) \rangle \psi_3^T P_+ \langle \psi_+(\tau_1) \psi_-(\tau_2) \rangle P_- \psi_3 \\ - 2 \langle T_-(\tau_1) T_+(\tau_2) \rangle \psi_3^T P_- \langle \psi_+(\tau_1) \psi_-(\tau_2) \rangle P_+ \psi_3 \} \quad (19)$$

Using our previous results for the propagators we obtain

$$S_{\text{so}} = -\frac{b}{T_0} \psi_3^T \gamma_1 \gamma_2 \psi_3 \int d\tau_1 d\tau_2 \{ 3 \Delta_B(\tau_1, \tau_2 | r^2) [\Delta_B(\tau_1, \tau_2 | r^2 + v) - \Delta_B(\tau_1, \tau_2 | r^2 - v)] \\ - \Delta_B(\tau_1, \tau_2 | r^2 + 2v) \Delta_B(\tau_1, \tau_2 | r^2 + v) + \Delta_B(\tau_1, \tau_2 | r^2 - 2v) \Delta_B(\tau_1, \tau_2 | r^2 - v) \}$$

It is evident that the terms linear and quadratic in velocity will cancel out in the above expression. To evaluate the v^3 term we need to expand out the propagators and compute the integrals. After doing the τ_2 integral the result will take the form

$$bv^3 \psi_3^T \gamma_1 \gamma_2 \psi_3 \int d\tau_1 \frac{1}{(b^2 + v^2 \tau_1^2)^{9/2}}.$$

Given this fact, it is easier to proceed by evaluating the τ_2 integral with $\tau_1 = 0$ and then restoring the τ_1 dependence afterwards. The expansion of the bosonic propagator is

$$\Delta_B(0, \tau_2 | b^2 + \alpha v) = \frac{e^{-b|\tau_2|}}{b} \left\{ \frac{1}{2} - \frac{\alpha v}{4b^2} [1 + b|\tau_2|] \right. \\ + \frac{v^2}{48b^4} [(9\alpha^2 - 6)(1 + b|\tau_2|) + (3\alpha^2 - 6)b^2|\tau_2|^2 - 4b^3|\tau_2|^3] \\ \left. - \frac{\alpha v^3}{96b^6} [(15\alpha^2 - 30)(1 + b|\tau_2|) + (6\alpha^2 - 24)b^2|\tau_2|^2 + (\alpha^2 - 14)b^3|\tau_2|^3 - 4b^4|\tau_2|^4] \right\} + \dots$$

Plugging this expansion into S_{so} and doing the τ_2 integral gives

$$S_{\text{so}} = -\frac{105}{32T_0} \int d\tau_1 \frac{bv^3 \psi_3^T \gamma_1 \gamma_2 \psi_3}{r^9}. \quad (20)$$

Now we can compare with the result from supergravity. Transforming back to Minkowski space, we find that the spin-orbit terms from (5) and (20) agree provided

$$\frac{(x^i J^{ij} v^j)(\vec{v} \cdot \vec{v})}{r^9} = -\frac{i}{2} \frac{bv^3 \psi_3^T \gamma_1 \gamma_2 \psi_3}{r^9}. \quad (21)$$

Does this equivalence make sense? To see that it does we need to recall the expression for the angular momentum operator of Matrix Theory. Starting from the action (7), the

operator which generates rotations in the transverse space is the sum of a bosonic piece and a fermionic piece. If we work in the rest frame of the source D0-brane - the one carrying the fermionic expectation value - then the bosonic contribution to the angular momentum of the source vanishes. The fermionic piece is the standard expression for the angular momentum of a spinor field,

$$J^{ij} = \frac{i}{2} \bar{\psi} \gamma_i \gamma_j \psi.$$

Recalling that ψ is Majorana, and that the relative velocity and separation of the D0-branes are along the x^1 and x^2 axes respectively, we find that (21) is satisfied. Thus we have verified that supergravity and Matrix Theory agree as to the leading spin dependence of the scattering amplitude.

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References

- [1] T. Banks, W. Fischler, S.H. Shenker, L. Susskind, *M Theory as a Matrix Model: A Conjecture*, *Phys. Rev.* **D55**, 5112 (1997), (hep-th/9610043).
- [2] M.R. Douglas, D. Kabat, P. Pouliot and S.H. Shenker, *D-branes and Short Distances in String Theory*, *Nucl. Phys.* **B485**, 85 (1997), (hep-th/9608024).
- [3] K. Becker and M. Becker, *A Two Loop Test of M(atr ix) theory*, hep-th/9705091.
- [4] K. Becker, M. Becker, J. Polchinski, and A. Tseytlin, *Higher Order Graviton Scattering in M(atr ix) Theory*, hep-th/9706072.
- [5] J.A. Harvey, *Spin Dependence of D0-brane Interactions*, hep-th/9706039.
- [6] J. Morales, C. Scrucca, and M. Serone, *A Note on Supersymmetric D-brane Dynamics*, hep-th/9709063
- [7] L. Susskind, *Another Conjecture About M(atr ix) theory*, hep-th/9704080.

- [8] M. Aganagic, C. Popescu, and J.H. Schwarz, *D-Brane Actions With Local Kappa Symmetry* *Phys. Lett.* **B393**, 311 (1997), hep-th/9610249; *Gauge Invariant and Gauge Fixed D-Brane Action*, *Nucl. Phys. B* **495**, 99 (1997), hep-th/9612080.
- [9] M. Cederwall, A. von Gussich, B. Nilsson, P. Sundell, and A. Westerberg, *The Dirichlet Super p-Branes in Ten-Dimensional Type IIA and IIB Supergravity*, *Nucl. Phys.* **B490**, 179 (1997) hep-th/9611159.
- [10] E. Bergshoeff and P.K. Townsend, *Super D-Branes* *Nucl. Phys.* **B490** 145 (1997), hep-th/961173
- [11] R.C. Myers and M.J. Perry, *Black Holes in Higher Dimensional Space-Times*, *Ann. Phys.* **172**, 304 (1986).
- [12] P. Berglund and D. Minic, *A Note on Effective Lagrangians in Matrix Theory*, hep-th/9708063.